

$A_{n-1}^{(1)}$ Reflection K-Matrices

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Abstract

We investigate the possible regular solutions of the boundary Yang-Baxter equation for the vertex models associated with the $A_{n-1}^{(1)}$ affine Lie algebra. We have classified them in two classes of solutions. The first class consists of $n(n-1)/2$ K-matrix solutions with three free parameters. The second class are solutions that depend on the parity of n . For n odd there exist n reflection K-matrices with $2 + [n/2]$ free parameters. It turns out that for n even there exist $n/2$ K-matrices with $2 + n/2$ free parameters and $n/2$ K-matrices with $1 + n/2$ free parameters.

PACS: 75.10.Jm; 05.90.+m

Keywords: Reflection equation, K-matrix.

February 8, 2008

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1 Introduction

The search for integrable models through the Yang-Baxter equation [1, 2, 3]

$$R_{12}(u-v)R_{13}(u)R_{23}(v) = R_{23}(v)R_{13}(u)R_{12}(u-v) \quad (1.1)$$

has been performed by the quantum group approach in [4]. Thus the problem is reduced to a linear one. Indeed, R matrices corresponding to vector representations of all non-exceptional affine Lie algebras were determined in this way in [5].

A similar approach is clearly desirable for finding solutions $K(u)$ of the boundary Yang-Baxter equation [6, 7]

$$R_{12}(u-v)K_1(u)R_{21}(u+v)K_2(v) = K_2(v)R_{12}(u+v)K_1(u)R_{21}(u-v). \quad (1.2)$$

With this goal in mind, the study of boundary quantum groups was initiated in [8]. However, as observed by Nepomechie [9], an independent systematic method of constructing the boundary quantum group generators is not yet available. In contrast to the bulk case [5], one cannot exploit boundary affine Toda field theory, since appropriated classical integrable boundary conditions are not yet known [10].

We are also sharing the hope that by studying the known examples of boundary quantum group generators, it may become possible to uncover their basic algebraic structure, and to find generalizations to all affine Lie algebras. Independent of the lack of an algebraic solution from the quantum group approach, there has been an increasing amount of effort towards the understanding of two-dimensional integrable theories with reflecting boundaries via solutions of the reflection equation (1.2). In field theory, attention is focused on the boundary S -matrix. In statistical mechanics, the emphasis has been on deriving solutions of (1.2) and the calculation of various surface critical phenomena, both at and away from criticality [11]. In condensed matter physics the actual target is the impurity problem.

The classification of all possible solutions of the reflection equation (1.2) by direct computation has been seen as a very difficult problem. However, recently we have proposed a method which allows the classification of the $D_{n+1}^{(2)}$ reflection K -matrices [12] as well as the K -matrices of the 19-vertex models [13]. In spite of these papers we decided to continue in this line in order to include the $A_{n-1}^{(1)}$ reflection K -matrices which will reveal us its algebraic structure.

We have organized this paper as follows. In Section 2 we choose the $A_{n-1}^{(1)}$ reflection equations and in Section 3 their solutions are derived and classified in two types. The last section is reserved for the conclusion. The first models have its K -matrices written explicitly in appendices.

2 The $A_{n-1}^{(1)}$ Reflection Equations

The R -matrix for the vertex models associated with the $A_{n-1}^{(1)}$ ($n \geq 2$) affine Lie algebra was originally found in the articles [14, 15] and as presented in [5] it has the form

$$\begin{aligned} R(u) = & a_1(u) \sum E_{ii} \otimes E_{ii} + a_2(u) \sum_{i \neq j} E_{ii} \otimes E_{jj} \\ & + a_3(u) \sum_{i < j} E_{ij} \otimes E_{ji} + a_4(u) \sum_{i > j} E_{ij} \otimes E_{ji}, \end{aligned} \quad (2.1)$$

where E_{ij} denotes the elementary n by n matrices ($(E_{ij})_{ab} = \delta_{ia}\delta_{jb}$) and the Boltzmann weights with functional dependence on the spectral parameter u are given by

$$a_1(u) = (e^u - q^2), \quad a_2(u) = q(e^u - 1), \quad a_3(u) = -(q^2 - 1), \quad a_4(u) = -e^u(q^2 - 1). \quad (2.2)$$

Here q denotes an arbitrary parameter.

For $n > 2$, the R -matrix (2.1) does not enjoy P and T symmetry but just PT invariance

$$\mathcal{P}_{12} R_{12}(u) \mathcal{P}_{12} \equiv R_{21}(u) = R_{12}(u)^{t_1 t_2} \quad (2.3)$$

and unitarity

$$R_{12}(u) R_{21}(-u) = \zeta(u) = a_1(u) a_1(-u). \quad (2.4)$$

It is not crossing invariant either but it obeys the weaker property [16]

$$\left\{ \left\{ \left\{ R_{12}(u)^{t_2} \right\}^{-1} \right\}^{t_2} \right\}^{-1} = \frac{\zeta(u + \rho)}{\zeta(u + 2\rho)} M_2 R_{12}(u + 2\rho) M_2^{-1}, \quad (2.5)$$

where M is a symmetry of the R -matrix

$$[R(u), M \otimes M] = 0, \quad M_{ij} = \delta_{ij} q^{n+1-2i}, \quad \rho = n \ln q. \quad (2.6)$$

The matrix $K_-(u)$ satisfies the left boundary Yang-Baxter equation, also known as the reflection equation,

$$R_{12}(u - v) \overset{1}{K}_-(u) R_{21}(u + v) \overset{2}{K}_-(v) = \overset{2}{K}_-(v) R_{12}(u + v) \overset{1}{K}_-(u) R_{21}(u - v), \quad (2.7)$$

which governs the integrability at boundary for a given bulk theory. A similar equation should also hold for the matrix $K_+(u)$ at the opposite boundary. However, for the $A_{n-1}^{(1)}$ models, one can see from [17] that the corresponding quantity

$$K_+(u) = K_-(-u - \rho)^t M, \quad (2.8)$$

satisfies the right boundary Yang-Baxter equation. Here $t = t_1 t_2$ and t_i stands for transposition taken in the i^{th} space.

Therefore, we can start for searching the matrices $K_-(u)$. In this paper only regular solutions will be considered, although there is much interest for non-regular $A_{n-1}^{(1)}$ solutions [18, 19].

Regular solutions mean that the matrix $K_-(u)$ has the form

$$K_-(u) = \sum_{i,j=1}^n k_{ij}(u) E_{ij} \quad (2.9)$$

and satisfies the condition

$$k_{ij}(0) = \delta_{ij}, \quad i, j = 1, 2, \dots, n. \quad (2.10)$$

Substituting (2.1) and (2.9) into (2.7), we will get n^4 functional equations for the k_{ij} matrix elements, many of which are dependent. In order to solve them, we shall proceed in the following way. First we consider the (i, j) component of the matrix equation (2.7). By differentiating it with respect to v and taking $v = 0$, we get algebraic equations involving the single variable u and n^2 parameters

$$\beta_{ij} = \left. \frac{dk_{ij}(v)}{dv} \right|_{v=0} \quad i, j = 1, 2, \dots, n. \quad (2.11)$$

Second, these algebraic equations are denoted by $E[i, j] = 0$ and collected into blocks $B[i, j]$, $i = 1, \dots, n^2 - i$ and $j = i, i + 1, \dots, n^2 - i$, defined by

$$B[i, j] = \begin{cases} E[i, j] = 0, & E[j, i] = 0, \\ E[n^2 + 1 - i, n^2 + 1 - j] = 0, & E[n^2 + 1 - j, n^2 + 1 - i] = 0. \end{cases} \quad (2.12)$$

For a given block $B[i, j]$, the equation $E[n^2 + 1 - i, n^2 + 1 - j] = 0$ can be obtained from the equation $E[i, j] = 0$ by interchanging

$$k_{ij} \longleftrightarrow k_{n+1-i, n+1-j}, \quad \beta_{ij} \longleftrightarrow \beta_{n+1-i, n+1-j}, \quad a_3(u) \leftrightarrow a_4(u) \quad (2.13)$$

and the equation $E[j, i] = 0$ is obtained from the equation $E[i, j] = 0$ by the interchanging

$$k_{ij} \longleftrightarrow k_{ji}, \quad \beta_{ij} \longleftrightarrow \beta_{ji} \quad (2.14)$$

In this way, we can control all equations and a particular solution is simultaneously connected with at least four equations.

3 The $A_{n-1}^{(1)}$ K-Matrix Solutions

Analyzing the $A_{n-1}^{(1)}$ reflection equations one can see that they possess a very special structure. Several equations exist involving only the elements out of the diagonal, k_{ij} ($i \neq j$), these are the simplest equations and we will solve them first.

By direct inspection one can see that the diagonal blocks $B[i, i]$ are uniquely solved by the relations

$$\beta_{ij}k_{ji}(u) = \beta_{ji}k_{ij}(u), \quad \forall i \neq j \quad (3.1)$$

It means that we only need to find the $n(n-1)/2$ elements k_{ij} ($i < j$). Now we choose a particular k_{ij} ($i < j$) to be different from zero, with $\beta_{ij} \neq 0$, and try to express all remaining elements in terms of this particular element. We have verified that this is possible provided that

$$k_{pq}(u) = \begin{cases} \frac{\alpha_4(u)}{\alpha_3(u)} \frac{\beta_{pq}}{\beta_{ij}} k_{ij}(u) & \text{if } p > i \text{ and } q > j \\ \frac{\beta_{pq}}{\beta_{ij}} k_{ij}(u) & \text{if } p > i \text{ and } q < j \end{cases}, \quad (p \neq q) \quad (3.2)$$

Combining (3.1) with (3.2) we will obtain a very strong entail for the elements out of the diagonal

$$k_{ij}(u) \neq 0 \Rightarrow \begin{cases} k_{pj}(u) = 0 & \text{for } p \neq i \\ k_{iq}(u) = 0 & \text{for } q \neq j \end{cases} \quad (3.3)$$

It means that for a given k_{ij} , the only elements different from zero in the i^{th} -row and in the j^{th} -column of $K_-(u)$ are k_{ii}, k_{ij}, k_{jj} and k_{ji} .

Analyzing more carefully these equations with the conditions (3.1) and (3.3), we have found from the $n(n-1)/2$ matrix elements k_{ij} ($i < j$) that there are two possibilities to choose a particular $k_{ij} \neq 0$:

- Only one non-diagonal element and its symmetric are allowed to be different from zero. Thus we have $n(n-1)/2$ reflection K -matrices with $n+2$ non-zero elements. Here we will denote by \mathbb{K}_{ij}^I ($i < j$), the K -matrix for which the non-diagonal element k_{ij} is the one chosen to be the non-zero matrix element. These matrices will be named Type-I solutions.
- For each $k_{ij} \neq 0$, additional non-diagonal elements and its asymmetric are allowed to be different from zero provided they satisfy the equations

$$k_{ij}(u)k_{ji}(u) = k_{rs}(u)k_{sr}(u) \quad \text{with } i + j = r + s \pmod n \quad (3.4)$$

It means that we will get n reflection K -matrices with the number of non-zero elements depending on the parity of n . Next, we choose the n possible particular elements as being k_{1j} , $j = 1, 2, \dots, n$ and k_{2n} . We will also denote the corresponding K -matrices by \mathbb{K}_{1j}^{II} , $j = 1, 2, \dots, n$ and \mathbb{K}_{2n}^{II} , respectively. These matrices are named Type-II solutions.

For example, the $A_2^{(1)}$ model has only Type-I solutions. The K -matrices are

$$\mathbb{K}_{12}^I = \begin{pmatrix} k_{11} & \mathbf{k}_{12} & 0 \\ k_{21} & k_{22} & 0 \\ 0 & 0 & k_{33} \end{pmatrix}, \mathbb{K}_{13}^I = \begin{pmatrix} k_{11} & 0 & \mathbf{k}_{13} \\ 0 & k_{22} & 0 \\ k_{31} & 0 & k_{33} \end{pmatrix}, \mathbb{K}_{23}^I = \begin{pmatrix} k_{11} & 0 & 0 \\ 0 & k_{22} & \mathbf{k}_{23} \\ 0 & k_{32} & k_{33} \end{pmatrix} \quad (3.5)$$

One can expect that these are the three possibilities to write the same solution for the $A_2^{(1)}$ model.

For the $A_3^{(1)}$ model we have six Type-I solutions $\{\mathbb{K}_{12}^I, \mathbb{K}_{13}^I, \mathbb{K}_{14}^I, \mathbb{K}_{23}^I, \mathbb{K}_{24}^I, \mathbb{K}_{34}^I\}$ all with six non-zero elements. In this model we also have two Type-II solutions $\{\mathbb{K}_{12}^{II}, \mathbb{K}_{14}^{II}\}$:

$$\mathbb{K}_{12}^{II} = \begin{pmatrix} k_{11} & \mathbf{k}_{12} & 0 & 0 \\ k_{21} & k_{22} & 0 & 0 \\ 0 & 0 & k_{33} & k_{34} \\ 0 & 0 & k_{43} & k_{44} \end{pmatrix} \quad \text{and} \quad \mathbb{K}_{14}^{II} = \begin{pmatrix} k_{11} & 0 & 0 & \mathbf{k}_{14} \\ 0 & k_{22} & k_{23} & 0 \\ 0 & k_{32} & k_{33} & 0 \\ k_{41} & 0 & 0 & k_{44} \end{pmatrix} \quad (3.6)$$

$$k_{12}k_{21} = k_{34}k_{43} \qquad k_{14}k_{41} = k_{23}k_{32}$$

For $n \geq 5$, in addition to the $n(n-1)/2$ Type-I solutions with $n+2$ non-zero matrix elements, we have also n Type-II solutions with the following property: if n is odd these K -matrices have $2n-1$ non-zero elements, but if n is even, half of these K -matrices have $2n$ non-zero elements and the remaining ones are matrices with $2n-2$ non-zero elements.

Although we already know as counting the K -matrices for the $A_{n-1}^{(1)}$ models we still have to identify among them which are similar. Indeed we can see a \mathbb{Z}_n similarity transformation which maps their matrix elements positions:

$$K^{(\alpha)} = h_\alpha K^{(0)} h_{n-\alpha}, \quad \alpha = 0, 1, 2, \dots, n-1 \quad (3.7)$$

where h_α are the \mathbb{Z}_n matrices

$$(h_\alpha)_{ij} = \delta_{i, i+\alpha} \mod n \quad (3.8)$$

In order to do this we can choose $K^{(0)}$ as \mathbb{K}_{12}^{II} and the similarity transformations (3.7) give us the $K^{(\alpha)}$ matrices whose matrix elements are in the same positions of the matrix elements of the \mathbb{K}_{1j}^{II} and \mathbb{K}_{2n}^{II} matrices. However, due to the fact that the relations (3.2) involve the ratio $\frac{a_4(u)}{a_3(u)} = e^u$, as well as the additional constraints (3.4), we could not find a similarity transformation among these \mathbb{K} 's matrices, even after a gauge transformation. Even for the Type-I solutions the similarity account is not simple due to the presence of three types of scalar functions and the constraint equations for the parameters β_{ij} . Nevertheless, as we have found a way to write all solutions, we can leave the similarity account to the reader.

Having identified these possibilities we may proceed in order to find the n diagonal elements $k_{ii}(u)$ in terms of the non-diagonal elements $k_{ij}(u)$ for each \mathbb{K}_{ij} matrix.

These procedure is now standard [13]. For instance, if we are looking for \mathbb{K}_{12}^{II} , the non-diagonal elements k_{ij} , $(i + j = 3 \pmod n)$ in terms of k_{12} are given by

$$k_{ij}(u) = \begin{cases} \frac{\beta_{ij}}{\beta_{12}} k_{12}(u) & \text{for } i + j = 3 \\ \frac{\beta_{ij}}{\beta_{12}} e^u k_{12}(u) & \text{for } i + j = 3 \pmod n \\ 0 & \text{otherwise} \end{cases} \quad (3.9)$$

for $i, j = 1, 2, \dots, n$, $(i \neq j)$.

Substituting (3.9) into the reflection equations we can now easily find the k_{ii} elements up to an arbitrary function, here identified as $k_{12}(u)$. Moreover, their consistency relations will yield us some constraints equations for the parameters β_{ij} .

After we have found all diagonal elements in terms of $k_{ij}(u)$, we can, without loss of generality, choose the arbitrary functions as

$$k_{ij}(u) = \frac{1}{2} \beta_{ij} (e^{2u} - 1), \quad i < j. \quad (3.10)$$

This choice allows us to work out the solutions in terms of the functions $f_{ii}(u)$ and $h_{ij}(u)$ defined by

$$f_{ii}(u) = \beta_{ii} (e^u - 1) + 1 \quad \text{and} \quad h_{ij}(u) = \frac{1}{2} \beta_{ij} (e^{2u} - 1), \quad (3.11)$$

for $i, j = 1, 2, \dots, n$.

Now, we will simply present the general solutions and write them explicitly for the first values of n in appendices. Let us start considering the Type-I solutions.

3.1 The Type-I K-Matrices

Here we have $n(n-1)/2$ reflection K -matrices with $n+2$ non-zero elements. For $1 < i < j \leq n$ we get $(n-2)(n-1)/2$ solutions

$$\begin{aligned} \mathbb{K}_{ij}^I = & f_{ii}(u)E_{ii} + e^{2u}f_{ii}(-u)E_{jj} + h_{ij}(u)E_{ij} + h_{ji}(u)E_{ji} \\ & + \mathcal{Z}_i(u) \sum_{l=1}^{i-1} E_{ll} + \mathcal{Y}_{i+1}^{(i)}(u) \sum_{l=i+1}^{j-1} E_{ll} + e^{2u} \mathcal{Z}_i(u) \sum_{l=j+1}^n E_{ll}, \end{aligned} \quad (3.12)$$

where $\mathcal{Z}_i(u)$ and $\mathcal{Y}_{i+1}^{(i)}(u)$ are scalar functions defined by

$$\mathcal{Z}_i(u) = f_{ii}(-u) + \frac{1}{2}(\beta_{ii} + \beta_{11})e^{-u}(e^{2u} - 1) \quad (3.13)$$

and

$$\mathcal{Y}_l^{(i)}(u) = f_{ii}(u) + \frac{1}{2}(\beta_{ll} - \beta_{ii})(e^{2u} - 1). \quad (3.14)$$

For $i = 1$ and $1 < j \leq n$ we get the $n-1$ remaining solutions

$$\begin{aligned} \mathbb{K}_{1j}^I = & f_{11}(u)E_{11} + e^{2u}f_{11}(-u)E_{jj} + h_{1j}(u)E_{1j} + h_{j1}(u)E_{j1} \\ & + \mathcal{Y}_2^{(1)}(u) \sum_{l=2}^{j-1} E_{ll} + \mathcal{X}_{j+1}(u) \sum_{l=j+1}^n E_{ll}, \end{aligned} \quad (3.15)$$

where a new scalar function appears,

$$\mathcal{X}_{j+1}(u) = e^{2u}f_{11}(-u) + \frac{1}{2}(\beta_{j+1,j+1} + \beta_{11} - 2)e^u(e^{2u} - 1). \quad (3.16)$$

The number of free parameters is fixed by the constraint equations which depend on the presence of these scalar functions: when $\mathcal{Y}_l^{(i)}(u)$ is present in \mathbb{K}_{ij} we have constraint equations of the type

$$\beta_{ij}\beta_{ji} = (\beta_{ll} + \beta_{ii} - 2)(\beta_{ll} - \beta_{ii}), \quad (3.17)$$

but, when $\mathcal{Z}_i(u)$ is present the corresponding constraints are of the type

$$\beta_{ij}\beta_{ji} = (\beta_{11} + \beta_{ii})(\beta_{11} - \beta_{ii}). \quad (3.18)$$

The presence of at least one $\mathcal{X}_{j+1}(u)$ yields a third type of constraints,

$$\beta_{ij}\beta_{ji} = (\beta_{j+1,j+1} + \beta_{11} - 2)(\beta_{j+1,j+1} - \beta_{11} - 2). \quad (3.19)$$

From (3.12) and (3.15) we can see that in each \mathbb{K}_{ij}^I we have at most two scalar functions. It means that all these \mathbb{K}_{ij}^I matrices are 3-parameter solutions of the reflection equation.

Finally, we observe that the solution with $i = 1$ and $j = n$, *i.e.*

$$\mathbb{K}_{1n}^I = f_{11}(u)E_{11} + e^{2u}f_{11}(-u)E_{nn} + h_{1n}(u)E_{1n} + h_{n1}(u)E_{n1} + \mathcal{Y}_2^{(1)}(u) \sum_{l=2}^{j-1} E_{ll} \quad (3.20)$$

has the constraint

$$\beta_{1n}\beta_{n1} = (\beta_{22} + \beta_{11} - 2)(\beta_{22} - \beta_{11}) \quad (3.21)$$

and, it is the solution derived by Abad and Rios [20].

3.2 The Type-II K-Matrices

Due to the property (3.4) we have found three Type-II general solutions for each $A_{n-1}^{(1)}$ model:

$$\begin{aligned} \text{Type - IIa} &= \{\mathbb{K}_{12p}^{II}\}, \quad \text{Type - IIb} = \{\mathbb{K}_{12p+1}^{II}\}, \quad \text{Type - IIc} = \mathbb{K}_{2n}^{II} \\ p &= 1, 2, \dots, [\frac{n}{2}] \end{aligned} \quad (3.22)$$

where $[\frac{n}{2}]$ being the integer part of $\frac{n}{2}$.

For n -odd, the Type-IIa solution is

$$\begin{aligned} \mathbb{K}_{12p}^{II} &= f_{11}(u) \sum_{j=1}^p E_{jj} + e^{2u}f_{11}(-u) \sum_{j=p+1}^{[\frac{n}{2}]+p} E_{jj} + e^{2u}f_{11}(u) \sum_{j=[\frac{n}{2}]+p+2}^n E_{jj} \\ &+ \mathcal{X}_{[\frac{n}{2}]+p+1}^{II}(u) E_{[\frac{n}{2}]+p+1, [\frac{n}{2}]+p+1} + \left(\sum_{\substack{i+j=1+2p \\ i \neq j}} + \sum_{\substack{i+j=1+2p \pmod n \\ i \neq j}} e^u \right) h_{ij}(u) E_{ij}, \end{aligned} \quad (3.23)$$

with constraint equations

$$\begin{aligned} \beta_{rs}\beta_{sr} &= \left(\beta_{[\frac{n}{2}]+p+1, [\frac{n}{2}]+p+1} + \beta_{11} - 2 \right) \left(\beta_{[\frac{n}{2}]+p+1, [\frac{n}{2}]+p+1} - \beta_{11} - 2 \right) \\ r + s &= 1 + 2p \pmod n. \end{aligned} \quad (3.24)$$

For the Type-IIb solutions we have obtained the following matrices

$$\begin{aligned}
\mathbb{K}_{12p+1}^{II} = & f_{11}(u) \sum_{j=1}^p E_{jj} + e^{2u} f_{11}(-u) \sum_{j=p+2}^{[\frac{n}{2}]+p+1} E_{jj} + e^{2u} f_{11}(u) \sum_{j=[\frac{n}{2}]+p+2}^n E_{jj} \\
& + \mathcal{X}_{p+1}(u) E_{p+1 \ p+1} + \left(\sum_{\substack{i+j=2+2p \\ i \neq j}} + \sum_{\substack{i+j=2+2p \bmod n \\ i \neq j}} e^u \right) h_{ij}(u) E_{ij},
\end{aligned} \tag{3.25}$$

together with their constraint equations

$$\begin{aligned}
\beta_{rs} \beta_{sr} &= (\beta_{p+1 \ p+1} + \beta_{11} - 2) (\beta_{p+1 \ p+1} - \beta_{11}) \\
r + s &= 2 + 2p \bmod n.
\end{aligned} \tag{3.26}$$

Finally, the Type-IIc solution is the matrix \mathbb{K}_{2n}^{II}

$$\begin{aligned}
\mathbb{K}_{2n}^{II} = & \mathcal{Z}_2(u) E_{11} + e^{2u} f_{22}(-u) \sum_{j=2}^{[\frac{n}{2}]+1} E_{jj} + e^{2u} f_{22}(u) \sum_{j=[\frac{n}{2}]+2}^n E_{jj} \\
& + \sum_{\substack{i+j=2 \bmod n \\ i \neq j}} h_{ij}(u) E_{ij},
\end{aligned} \tag{3.27}$$

for which the constraint equations are

$$\begin{aligned}
\beta_{rs} \beta_{sr} &= (\beta_{11} + \beta_{22}) (\beta_{11} - \beta_{22}) \\
r + s &= 2 \bmod n.
\end{aligned} \tag{3.28}$$

The function $\mathcal{Z}_2(u)$ is given by (3.13) and the functions $\mathcal{X}_j(u)$ by (3.16), while the functions $f_{11}(u)$, $f_{22}(u)$ and $h_{ij}(u)$ are given by (3.11). Therefore we have n reflection K -matrices for the $A_{n-1}^{(1)}$ models (n odd). They are $(2 + [\frac{n}{2}])$ -free parameter solutions with $2n - 1$ non-zero matrix elements.

When n is even we have a similar identification but substantial differences exist.

In the n -even case the Type-IIa solutions are the matrices

$$\begin{aligned}
\mathbb{K}_{12p}^{II} = & f_{11}(u) \sum_{j=1}^p E_{jj} + e^{2u} f_{11}(-u) \sum_{j=p+1}^{\frac{n}{2}+p} E_{jj} + e^{2u} f_{11}(u) \sum_{j=\frac{n}{2}+p+1}^n E_{jj} \\
& + \left(\sum_{\substack{i+j=2 \\ i \neq j}} + \sum_{\substack{i+j=1+2p \bmod n \\ i \neq j}} e^u \right) h_{ij}(u) E_{ij},
\end{aligned} \tag{3.29}$$

with constraint equations

$$\beta_{12p}\beta_{2p1} = \beta_{rs}\beta_{sr}, \quad r + s = 1 + 2p \pmod{n}. \quad (3.30)$$

For the Type-IIb solutions we have

$$\begin{aligned} \mathbb{K}_{12p+1}^{II} &= f_{11}(u) \sum_{j=1}^p E_{jj} + \mathcal{Y}_{p+1}^{(1)}(u) E_{p+1 \ p+1} + e^{2u} f_{11}(-u) \sum_{j=p+2}^{\frac{n}{2}+p} E_{jj} \\ &\quad + \mathcal{X}_{\frac{n}{2}+p+1}(u) E_{\frac{n}{2}+p+1 \ \frac{n}{2}+p+1} + e^{2u} f_{11}(u) \sum_{j=\frac{n}{2}+p+2}^n E_{jj} \\ &\quad + \left(\sum_{\substack{i+j=2+2p \\ i \neq j}} + \sum_{\substack{i+j=2+2p \pmod{n} \\ i \neq j}} e^u \right) h_{ij}(u) E_{ij}, \end{aligned} \quad (3.31)$$

with the following constraint equations

$$\begin{aligned} \beta_{rs}\beta_{sr} &= (\beta_{p+1 \ p+1} + \beta_{11} - 2) (\beta_{p+1 \ p+1} - \beta_{11}) \\ &= (\beta_{\frac{n}{2}+p+1 \ \frac{n}{2}+p+1} + \beta_{11} - 2) (\beta_{\frac{n}{2}+p+1 \ \frac{n}{2}+p+1} - \beta_{11} - 2), \\ r + s &= 2 + 2p \pmod{n}, \end{aligned} \quad (3.32)$$

Again, the Type-IIc solution is the matrix \mathbb{K}_{2n}^{II}

$$\begin{aligned} \mathbb{K}_{2n}^{II} &= \mathcal{Z}_2(u) E_{11} + f_{22}(u) \sum_{j=2}^{\frac{n}{2}} E_{jj} + \mathcal{Y}_{\frac{n}{2}+1}^{(2)}(u) E_{\frac{n}{2}+1 \ \frac{n}{2}+1} \\ &\quad + e^{2u} f_{22}(-u) \sum_{j=\frac{n}{2}+2}^n E_{jj} + \sum_{\substack{i+j=2 \pmod{n} \\ i \neq j}} h_{ij}(u) E_{ij}, \end{aligned} \quad (3.33)$$

with the constraint equations

$$\begin{aligned} \beta_{rs}\beta_{sr} &= (\beta_{11} - \beta_{22}) (\beta_{11} - \beta_{22}) \\ &= \left(\beta_{\frac{n}{2}+1 \ \frac{n}{2}+1} + \beta_{22} - 2 \right) \left(\beta_{\frac{n}{2}+1 \ \frac{n}{2}+1} - \beta_{22} \right), \\ r + s &= 2 \pmod{n}. \end{aligned} \quad (3.34)$$

where the scalar functions $\mathcal{Z}_2(u)$ and $\mathcal{Y}_{p+1}^{(2)}(u)$ are given by (3.13) and (3.14), respectively.

Here we observe that for n even, the Type-IIa is a $(2 + \frac{n}{2})$ -free parameter solution with $2n$ non-zero matrix elements, while the Type-IIb and the Type-IIc are $(1 + \frac{n}{2})$ -free parameter solutions with $2(n - 1)$ non-zero matrix elements.

4 Conclusion

The absence of an algebraic method such as the quantum group approaches leads us to believe that a direct computation from their reflection equations should be a starting point to obtain its classification.

After a systematic study of the functional equations we find that there are two types of solutions for the $A_{n-1}^{(1)}$ models. We call of Type-I the K -matrices with three free parameters and $n + 2$ non-zero matrix elements. These solutions were denoted by \mathbb{K}_{ij}^I to emphasize the non-zero element out of the diagonal and its symmetric, which results in $n(n - 1)/2$ reflection K -matrices.

The Type-II solutions are more interesting because their have many free parameters. The $A_{n-1}^{(1)}$ models for n odd, in addition to the Type-I solutions, have n Type-II solutions with $2n - 1$ non-zero matrix elements and $(2 + [\frac{n}{2}])$ free parameters. It turns out that for n even we also have n Type-II solutions but half of them are K -matrices with $2n$ non-zero matrix elements and $(2 + \frac{n}{2})$ free parameters, while the remaining ones have $2(n - 1)$ non-zero matrix elements with $(1 + \frac{n}{2})$ free parameters.

The corresponding $K_+(u)$ are obtained from the isomorphism (2.8). Out of this classification we have the trivial solution ($K_- = 1, K_+ = M$) for these models. Thus we ended our discussion on the reflection matrices for the vertex models associated with the $A_{n-1}^{(1)}$ affine Lie algebra.

To complete the classification for all non-exceptional Lie algebras we still have to consider the vertex models associated with the $B_n^{(1)}$, $C_n^{(1)}$, $D_n^{(1)}$, $A_{2n}^{(2)}$ and $A_{2n-1}^{(2)}$ Lie algebras.

Acknowledgment: This work was supported in part by Fundação de Amparo à Pesquisa do Estado de São Paulo–FAPESP–Brasil and by Conselho Nacional de Desenvolvimento–CNPq–Brasil.

A The $A_1^{(1)}$ Reflection K-Matrices

This is a very special case among the $A_{n-1}^{(1)}$ models. We note that there is only one general K -matrix with 4 non-zero matrix elements [21, 22]. From the Type-IIa solutions (3.29) or from the Type-I solutions (3.15) it is the \mathbb{K}_{12} matrix

$$\mathbb{K}_{12}^I = \begin{pmatrix} f_{11}(u) & h_{12}(u) \\ h_{21}(u) & e^{2u} f_{11}(-u) \end{pmatrix}$$

Although there is no constraint equation in this case, the regular condition (2.10) has fixed in three the number of free parameters, in agreement with all Type-I reflection K -matrices.

B The $A_2^{(1)}$ Reflection K-Matrices

This is also a special case because it has only the Type-I solutions \mathbb{K}_{12}^I , \mathbb{K}_{13}^I and \mathbb{K}_{23}^I . From (3.15) we have

$$\begin{aligned}\mathbb{K}_{12}^I &= f_{11}(u)E_{11} + e^{2u}f_{11}(-u)E_{22} + h_{12}(u)E_{12} + h_{21}(u)E_{21} + \mathcal{X}_3(u)E_{33} \\ &= \begin{pmatrix} f_{11}(u) & h_{12}(u) & 0 \\ h_{21}(u) & e^{2u}f_{11}(-u) & 0 \\ 0 & 0 & \mathcal{X}_3(u) \end{pmatrix},\end{aligned}\tag{B.1}$$

with the four parameters $\beta_{11}, \beta_{12}, \beta_{21}$ and β_{33} satisfied the constraint equation

$$\beta_{12}\beta_{21} = (\beta_{33} - \beta_{11} - 2)(\beta_{33} + \beta_{11} - 2).\tag{B.2}$$

Two diagonal solutions are derived from (B.1) due to this constraint equation

$$\begin{aligned}\lim_{\beta_{33} \rightarrow -\beta_{11}+2} \mathcal{X}_3(u) &= e^{2u}f_{11}(-u) \\ \Rightarrow D_1 &= \text{diag}(f_{11}(u), e^{2u}f_{11}(-u), e^{2u}f_{11}(-u))\end{aligned}\tag{B.3}$$

and

$$\begin{aligned}\lim_{\beta_{33} \rightarrow \beta_{11}+2} \mathcal{X}_3(u) &= e^{2u}f_{11}(u) \\ \Rightarrow D_2 &= \text{diag}(f_{11}(u), e^{2u}f_{11}(-u), e^{2u}f_{11}(u))\end{aligned}\tag{B.4}$$

The matrix \mathbb{K}_{13}^I is also given by (3.15)

$$\begin{aligned}\mathbb{K}_{13}^I &= f_{11}(u)E_{11} + e^{2u}f_{11}(-u)E_{33} + h_{13}(u)E_{13} + h_{31}(u)E_{31} + \mathcal{Y}_2^{(1)}(u)E_{22} \\ &= \begin{pmatrix} f_{11}(u) & 0 & h_{13}(u) \\ 0 & \mathcal{Y}_2^{(1)}(u) & 0 \\ h_{31}(u) & 0 & e^{2u}f_{11}(-u) \end{pmatrix},\end{aligned}\tag{B.5}$$

but now the constraint equation is

$$\beta_{13}\beta_{31} = (\beta_{22} + \beta_{11} - 2)(\beta_{22} - \beta_{11}),\tag{B.6}$$

and the corresponding diagonal reductions are

$$\begin{aligned}\lim_{\beta_{22} \rightarrow -\beta_{11}+2} \mathcal{Y}_2^{(1)}(u) &= e^{2u}f_{11}(-u) \\ D_3 &= \text{diag}(f_{11}(u), e^{2u}f_{11}(-u), e^{2u}f_{11}(-u))\end{aligned}\tag{B.7}$$

and

$$\begin{aligned}\lim_{\beta_{22} \rightarrow \beta_{11}} \mathcal{Y}_2^{(1)}(u) &= f_{11}(u) \\ D_4 &= \text{diag}(f_{11}(u), f_{11}(u), e^{2u}f_{11}(-u))\end{aligned}\tag{B.8}$$

For the solution named \mathbb{K}_{23}^I we recall the equation (3.12) with $i = 2$ and $j = 3$

$$\begin{aligned}\mathbb{K}_{23}^I &= f_{22}(u)E_{22} + e^{2u}f_{ii}(-u)E_{33} + h_{23}(u)E_{23} + h_{32}(u)E_{32} + \mathcal{Z}_2(u)E_{11} \\ &= \begin{pmatrix} \mathcal{Z}_2(u) & 0 & 0 \\ 0 & f_{22}(u) & h_{23}(u) \\ 0 & h_{32}(u) & e^{2u}f_{22}(-u) \end{pmatrix},\end{aligned}\tag{B.9}$$

with the constraint

$$\beta_{23}\beta_{32} = (\beta_{11} + \beta_{22})(\beta_{11} - \beta_{22})\tag{B.10}$$

we get more two diagonal solutions

$$\begin{aligned}\lim_{\beta_{22} \rightarrow -\beta_{11}} \mathcal{Z}_2(u) &= f_{22}(-u) \\ D_5 &= \text{diag}(f_{22}(-u), f_{22}(u), e^{2u}f_{22}(-u))\end{aligned}\tag{B.11}$$

and

$$\begin{aligned}\lim_{\beta_{22} \rightarrow \beta_{11}} \mathcal{Z}_2(u) &= f_{22}(u) \\ D_6 &= \text{diag}(f_{22}(u), f_{22}(u), e^{2u}f_{22}(-u))\end{aligned}\tag{B.12}$$

Due to the constraint equations these reflection K -matrices have only three free parameters and the corresponding diagonal solutions have only one free parameter.

Here we observe that only four of these diagonal solutions are independents because $D_6 = D_4$ and $D_3 = D_1$. Here we also note that the solutions D_1 and D_4 are the diagonal solutions derived by the first time in [21] and \mathbb{K}_{13}^I is the non-diagonal solution derived in [20].

In a certain sense these solution are particular because they do not reveal us all properties shared by the regular $A_{n-1}^{(1)}$ reflection K -matrices for n odd. Before we consider the next odd case, let us consider the case $n = 4$.

C The $A_3^{(1)}$ Reflection K-Matrices

In this case the structure of the general solution begins to appear but it is still particular because half of the Type-II solutions are Type-I solutions.

The \mathbb{K}_{1j} matrices for the Type-I solutions are given by (3.15). For \mathbb{K}_{12}^I we get

$$\mathbb{K}_{12}^I = \begin{pmatrix} f_{11}(u) & h_{12}(u) & 0 & 0 \\ h_{21}(u) & e^{2u}f_{11}(-u) & 0 & 0 \\ 0 & 0 & \mathcal{X}_3(u) & 0 \\ 0 & 0 & 0 & \mathcal{X}_3(u) \end{pmatrix}\tag{C.1}$$

with the constraint

$$\beta_{12}\beta_{21} = (\beta_{33} + \beta_{11} - 2)(\beta_{33} - \beta_{11} - 2) \quad (\text{C.2})$$

For \mathbb{K}_{13}^I we have

$$\mathbb{K}_{13}^I = \begin{pmatrix} f_{11}(u) & 0 & h_{13}(u) & 0 \\ 0 & \mathcal{Y}_2^{(1)}(u) & 0 & 0 \\ h_{31}(u) & 0 & e^{2u}f_{11}(-u) & 0 \\ 0 & 0 & 0 & \mathcal{X}_4(u) \end{pmatrix} \quad (\text{C.3})$$

with the constraint

$$\beta_{13}\beta_{31} = (\beta_{44} + \beta_{11} - 2)(\beta_{44} - \beta_{11} - 2) = (\beta_{22} + \beta_{11} - 2)(\beta_{22} - \beta_{11}) \quad (\text{C.4})$$

The \mathbb{K}_{14}^I matrix is

$$\mathbb{K}_{14}^I = \begin{pmatrix} f_{11}(u) & 0 & 0 & h_{14}(u) \\ 0 & \mathcal{Y}_2^{(1)}(u) & 0 & 0 \\ 0 & 0 & \mathcal{Y}_2^{(1)}(u) & 0 \\ h_{41}(u) & 0 & 0 & e^{2u}f_{11}(-u) \end{pmatrix} \quad (\text{C.5})$$

with

$$\beta_{14}\beta_{41} = (\beta_{22} + \beta_{11} - 2)(\beta_{22} - \beta_{11}) \quad (\text{C.6})$$

The remaining Type-I K -matrices are given by (3.12). For \mathbb{K}_{23}^I we get

$$\mathbb{K}_{23}^I = \begin{pmatrix} \mathcal{Z}_2(u) & 0 & 0 & 0 \\ 0 & f_{22}(u) & h_{23}(u) & 0 \\ 0 & h_{32}(u) & e^{2u}f_{22}(-u) & 0 \\ 0 & 0 & 0 & e^{2u}\mathcal{Z}_2(u) \end{pmatrix} \quad (\text{C.7})$$

with constraint

$$\beta_{23}\beta_{32} = (\beta_{11} + \beta_{22})(\beta_{11} - \beta_{22}) \quad (\text{C.8})$$

For \mathbb{K}_{24}^I we have

$$\mathbb{K}_{24}^I = \begin{pmatrix} \mathcal{Z}_2(u) & 0 & 0 & 0 \\ 0 & f_{22}(u) & 0 & h_{24}(u) \\ 0 & 0 & \mathcal{Y}_3^{(2)}(u) & 0 \\ 0 & h_{42}(u) & 0 & e^{2u}f_{22}(-u) \end{pmatrix} \quad (\text{C.9})$$

with the constraint

$$\beta_{24}\beta_{42} = (\beta_{11} + \beta_{22})(\beta_{11} - \beta_{22}) = (\beta_{33} + \beta_{22} - 2)(\beta_{33} - \beta_{22}) \quad (\text{C.10})$$

and finally for \mathbb{K}_{34}^I

$$\mathbb{K}_{34}^I = \begin{pmatrix} \mathcal{Z}_3(u) & 0 & 0 & 0 \\ 0 & \mathcal{Z}_3(u) & 0 & 0 \\ 0 & 0 & f_{33}(u) & h_{34}(u) \\ 0 & 0 & h_{34}(u) & e^{2u}f_{33}(-u) \end{pmatrix} \quad (\text{C.11})$$

with

$$\beta_{34}\beta_{43} = (\beta_{11} + \beta_{33})(\beta_{11} - \beta_{33}) \quad (\text{C.12})$$

For the Type-IIa solutions we get from (3.29) more two K -matrices

$$\mathbb{K}_{12}^{II} = \begin{pmatrix} f_{11}(u) & h_{12}(u) & 0 & 0 \\ h_{21}(u) & e^{2u}f_{11}(-u) & 0 & 0 \\ 0 & 0 & e^{2u}f_{11}(-u) & e^u h_{34}(u) \\ 0 & 0 & e^u h_{43}(u) & e^{2u}f_{11}(u) \end{pmatrix} \quad (\text{C.13})$$

with the eight non-zero elements satisfying a constraint equation

$$\beta_{12}\beta_{21} = \beta_{34}\beta_{43}. \quad (\text{C.14})$$

The another K -matrix is given by

$$\mathbb{K}_{14}^{II} = \begin{pmatrix} f_{11}(u) & 0 & 0 & h_{14}(u) \\ 0 & f_{11}(u) & h_{23}(u) & 0 \\ 0 & h_{32}(u) & e^{2u}f_{11}(-u) & 0 \\ h_{41}(u) & 0 & 0 & e^{2u}f_{11}(-u) \end{pmatrix} \quad (\text{C.15})$$

with the a constraint equation

$$\beta_{14}\beta_{41} = \beta_{23}\beta_{32} \quad (\text{C.16})$$

Note that both Type-II solutions (C.13) and (C.15) have four free parameters.

Next, we can solve these constraint equations to derive eighteen diagonal solutions. Using the following reductions for the scalar functions $\mathcal{X}_{j+1}(u)$, $\mathcal{Y}_l^{(i)}(u)$ and $\mathcal{Z}_i(u)$

$$\begin{aligned} \lim_{\beta_{j+1} \rightarrow -\beta_{11}+2} \mathcal{X}_{j+1}(u) &= e^{2u}f_{11}(-u), & \lim_{\beta_{j+1} \rightarrow \beta_{11}+2} \mathcal{X}_{j+1}(u) &= e^{2u}f_{11}(u), \\ \lim_{\beta_{ll} \rightarrow -\beta_{ii}+2} \mathcal{Y}_l^{(i)}(u) &= e^{2u}f_{ii}(-u), & \lim_{\beta_{ll} \rightarrow \beta_{ii}} \mathcal{Y}_l^{(i)}(u) &= f_{ii}(u), \\ \lim_{\beta_{11} \rightarrow -\beta_{ii}} \mathcal{Z}_i(u) &= f_{ii}(-u), & \lim_{\beta_{11} \rightarrow \beta_{ii}} \mathcal{Z}_i(u) &= f_{ii}(u). \end{aligned} \quad (\text{C.17})$$

we can see that only half of these diagonal solutions are independents:

$$\begin{aligned}
D_1 &= \text{diag}(f(u), e^{2u}f(-u), e^{2u}f(-u), e^{2u}f(-u)), \\
D_2 &= \text{diag}(f(u), e^{2u}f(-u), e^{2u}f(u), e^{2u}f(u)), \\
D_3 &= \text{diag}(f(u), f(u), e^{2u}f(-u), e^{2u}f(-u)), \\
D_4 &= \text{diag}(f(u), e^{2u}f(-u), e^{2u}f(-u), e^{2u}f(u)), \\
D_5 &= \text{diag}(f(u), f(u), e^{2u}f(-u), e^{2u}f(u)), \\
D_6 &= \text{diag}(f(u), f(u), f(u), e^{2u}f(-u)), \\
D_7 &= \text{diag}(f(-u), f(u), e^{2u}f(-u), e^{2u}f(-u)), \\
D_8 &= \text{diag}(f(-u), f(u), f(u), e^{2u}f(-u)), \\
D_9 &= \text{diag}(f(-u), f(-u), f(u), e^{2u}f(-u)),
\end{aligned} \tag{C.18}$$

where we have used a compact notation for the functions $f_{ii}(u)$

$$f_{ii}(u) \equiv f(u) = \beta(e^u - 1) + 1 \tag{C.19}$$

where β is the free parameter.

D The $A_4^{(1)}$ Type-II Reflection K-Matrices

Here we will only write explicitly the five Type-II solutions and their constraint equations for the $A_4^{(1)}$ model. They have nine non-zero matrix elements and four free parameters:

$$\begin{aligned}
\mathbb{K}_{12}^{II} &= \begin{pmatrix} f_{11}(u) & h_{12}(u) & 0 & 0 & 0 \\ h_{21}(u) & e^{2u}f_{11}(-u) & 0 & 0 & 0 \\ 0 & 0 & e^{2u}f_{11}(-u) & 0 & e^uh_{35}(u) \\ 0 & 0 & 0 & \mathcal{X}_4(u) & 0 \\ 0 & 0 & e^uh_{53}(u) & 0 & e^{2u}f_{11}(u) \end{pmatrix}, \\
\beta_{12}\beta_{21} &= \beta_{35}\beta_{53} = (\beta_{44} + \beta_{11} - 2)(\beta_{44} - \beta_{11} - 2),
\end{aligned} \tag{D.1}$$

$$\begin{aligned}
\mathbb{K}_{14}^{II} &= \begin{pmatrix} f_{11}(u) & 0 & 0 & h_{14}(u) & 0 \\ 0 & f_{11}(u) & h_{23}(u) & 0 & 0 \\ 0 & h_{32}(u) & e^{2u}f_{11}(-u) & 0 & 0 \\ h_{41}(u) & 0 & 0 & e^{2u}f_{11}(-u) & 0 \\ 0 & 0 & 0 & 0 & \mathcal{X}_5(u) \end{pmatrix}, \\
\beta_{14}\beta_{41} &= \beta_{23}\beta_{32} = (\beta_{55} + \beta_{11} - 2)(\beta_{55} - \beta_{11} - 2),
\end{aligned} \tag{D.2}$$

$$\mathbb{K}_{13}^{II} = \begin{pmatrix} f_{11}(u) & 0 & h_{13}(u) & 0 & 0 \\ 0 & \mathcal{X}_2(u) & 0 & 0 & 0 \\ h_{31}(u) & 0 & e^{2u} f_{11}(-u) & 0 & 0 \\ 0 & 0 & 0 & e^{2u} f_{11}(-u) & e^u h_{45}(u) \\ 0 & 0 & 0 & e^u h_{54}(u) & e^{2u} f_{11}(u) \end{pmatrix},$$

$$\beta_{13}\beta_{31} = \beta_{45}\beta_{54} = (\beta_{22} + \beta_{11} - 2)(\beta_{22} - \beta_{11}), \quad (\text{D.3})$$

$$\mathbb{K}_{15}^{II} = \begin{pmatrix} f_{11}(u) & 0 & 0 & 0 & h_{15}(u) \\ 0 & f_{11}(u) & 0 & h_{24}(u) & 0 \\ 0 & 0 & \mathcal{X}_3(u) & 0 & 0 \\ 0 & h_{42}(u) & 0 & e^{2u} f_{11}(-u) & 0 \\ h_{51}(u) & 0 & 0 & 0 & e^{2u} f_{11}(-u) \end{pmatrix},$$

$$\beta_{15}\beta_{51} = \beta_{24}\beta_{42} = (\beta_{33} + \beta_{11} - 2)(\beta_{33} - \beta_{11}), \quad (\text{D.4})$$

$$\mathbb{K}_{25}^{II} = \begin{pmatrix} \mathcal{Z}_2(u) & 0 & 0 & 0 & 0 \\ 0 & f_{11}(u) & 0 & 0 & h_{25}(u) \\ 0 & 0 & f_{11}(u) & h_{34}(u) & 0 \\ 0 & 0 & h_{43}(u) & e^{2u} f_{11}(-u) & 0 \\ 0 & h_{52}(u) & 0 & 0 & e^{2u} f_{11}(-u) \end{pmatrix},$$

$$\beta_{25}\beta_{52} = \beta_{34}\beta_{43} = (\beta_{11} + \beta_{22})(\beta_{11} - \beta_{22}). \quad (\text{D.5})$$

The corresponding diagonal solutions are also one-parameter solutions.

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